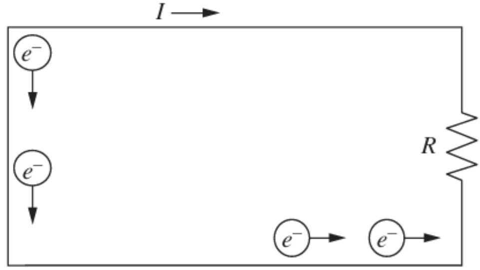
—Chapter 7—

Faraday's Induction

7-1 Electromotive Force

A. CURRENT FLOW THROUGH THE CONDUCTOR

(1) Considering a circuit contains the steady current I.



The power dissipated in a resistor is given by

$$P = I^2 R$$

The current will decrease to zero eventually.

(2) Assume Ohm's law holds:

$$\vec{J} = \sigma \vec{E}$$

For the loop, we have

$$\oint_{\mathcal{C}} \vec{J} \cdot d\vec{s} = \sigma \oint_{\mathcal{C}} \vec{E} \cdot d\vec{s}$$

Then using Stokes' theorem,

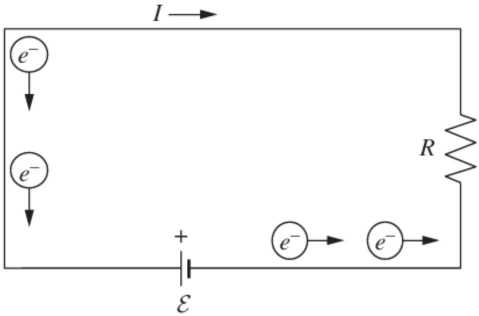
$$\oint_{\mathcal{C}} \vec{E} \cdot d\vec{s} = \int_{\mathcal{S}} \underbrace{\left(\nabla \times \vec{E}\right)}_{\substack{=0 \\ \text{for} \\ \text{electrostatic} \\ \text{field}}} \cdot d\vec{a} = 0$$

we obtain

$$\oint_{\mathcal{C}} \vec{J} \cdot d\vec{s} = 0$$

Therefore, I must be zero for the loop.

(3) To keep the current flowing through the circuit there must be a force pushing the charges through the circuit.



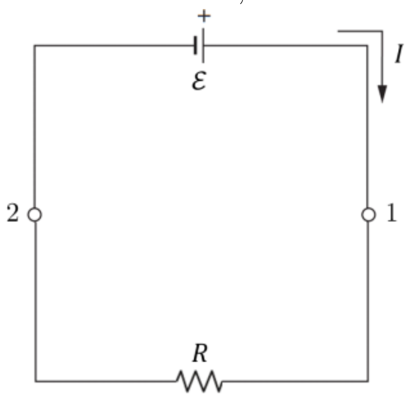
The battery is a source to continuously provide a non-electrostatic field to push charges moving through the circuit. The source is called an electromotive force (EMF) on the circuit. [c.f.4-3]

B. ELECTROMOTIVE FORCE

(1) There are two forces involved in driving current around a circuit: the source and an electrostatic force. We represent the effect of any source of EMF by a non-electrostatic field $\vec{E}'$. Thus, Ohm's law becomes

$$\vec{J} = \sigma\left(\vec{E} + \vec{E}'\right)$$

(2) For a closed circuit,



$$\oint_{C} \vec{J} \cdot d\vec{s} = \sigma \oint_{C} \vec{E} \cdot d\vec{s} + \sigma \oint_{C} \vec{E}' \cdot d\vec{s}$$

Since

$$\oint_{C} \vec{E} \cdot d\vec{s} = 0 \cdots\cdots \text{for electrostatic field}$$

we obtain

$$\oint_{\mathcal{C}} \vec{E}' \cdot d\vec{s} = \frac{1}{\sigma} \oint_{\mathcal{C}} \vec{J} \cdot d\vec{s}$$

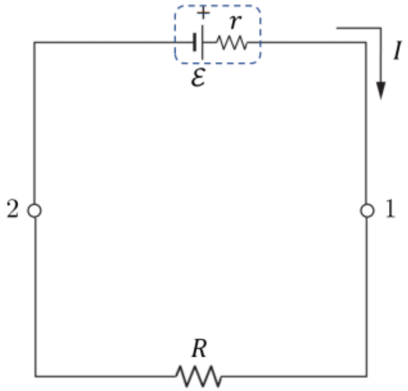
If the wire have a uniform cross-section A, then we have $J = I/A$. Thus, we obtain

$$\frac{1}{\sigma} \oint_{\mathcal{C}} \vec{J} \cdot d\vec{s} = \frac{1}{\sigma} \oint_{\mathcal{C}} \frac{I}{A} \hat{n} \cdot d\vec{s} = \overbrace{\left(\frac{L}{\sigma A}\right) I = \underbrace{RI = \mathcal{E}}_{\text{voltage}}}^{\text{[c.f.4-2]}}$$

We then define the EMF as

$$\mathcal{E} = \oint_{\mathcal{C}} \vec{E}' \cdot d\vec{s} = \frac{1}{q} \oint_{\mathcal{C}} \vec{f} \cdot d\vec{s}$$

(3) Since in practice the potential difference V between terminal 1 and terminal 2 will drop only a little below the value $\mathcal{E}$, r plays the role of an effective "internal resistance" which accounts for loss mechamisms within the source itself, i.e.,

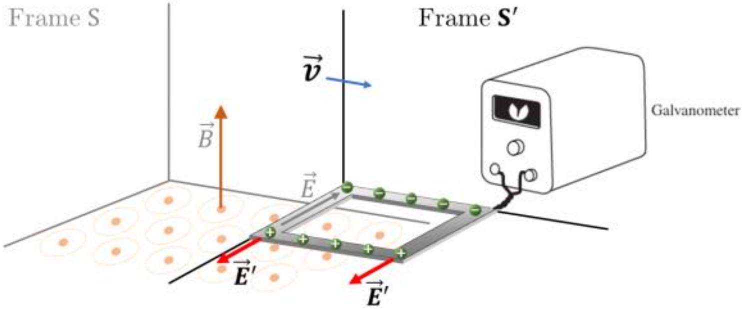


$$\mathcal{E} - Ir = \underbrace{V = IR}_{\text{voltage}}$$

7-2 Faraday's Law

A. INDUCED EMF

(1) A rectangular loop of wire is moving at constant velocity $\vec{v}$ in a uniform magnetic field $\vec{B}$.



In frame S:

Any charged particle that is carried along with the loop experiences a force

$$\vec{f} = q\vec{v} \times \vec{B}$$

The force $\vec{f}$ pushes negative charges toward one end of the loop, leaving the other end positively charged. This goes on until these separated charges themselves cause an electric field $\vec{E}$ such that, everywhere in the interior of the loop,

$$q\vec{E} = -\vec{f}$$

Evaluate the line integral of $\vec{f}$, taken around the whole loop (counterclockwise as viewed from above).

$$\oint_C \vec{f} \cdot d\vec{s} = qvBw - qvBw = 0$$

The electromotive force that causes charge to circulate around a closed path is

$$\mathcal{E} = \frac{1}{q} \oint_C \vec{f} \cdot d\vec{s} = 0$$

⇒ The galvanometer *does not* deflect.

In frame S′:

According to equations in 5-5-A(a), if there exists a frame in which $\vec{E} = 0$, then in any other frame, we have

$$\left.\begin{array}{ll}\vec{E}'_{\parallel}=0, & \vec{E}'_{\perp}=\gamma\vec{v}\times\vec{B}_{\perp}\\ \vec{B}'_{\parallel}=\vec{B}_{\parallel}, & \vec{B}'_{\perp}=\gamma\vec{B}_{\perp}\end{array}\right\}\Rightarrow \vec{E}'_{\perp}=\vec{v}\times\vec{B}'_{\perp}\Rightarrow \vec{E}'=\vec{v}\times\vec{B}'$$

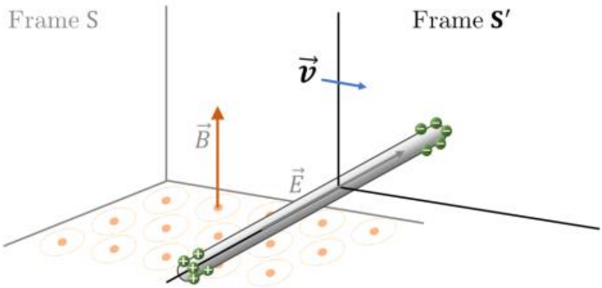
The redistribution of charge on the surface of the loop so as to make the electric field opposite the direction of $\vec{E}'$ and cancel the field inside the loop. Thus, the net field inside the loop is zero.

$$\mathcal{E}=\int_{\mathcal{C}}\underbrace{\vec{E}'_{\text{net}}}_{=0}\cdot d\vec{s}'=0$$

$\Rightarrow$ The galvanometer *does not* deflect.

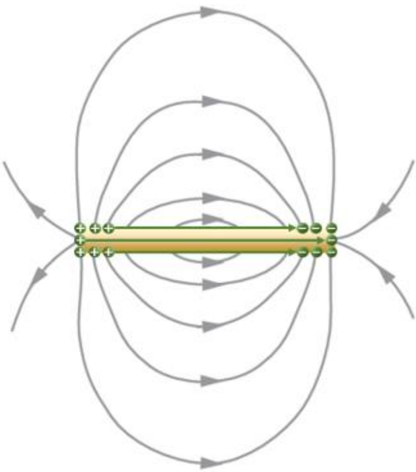
EXAMPLES:

1. A metal rod is moving at constant velocity $\vec{v}$ in a uniform magnetic field $\vec{B}$.



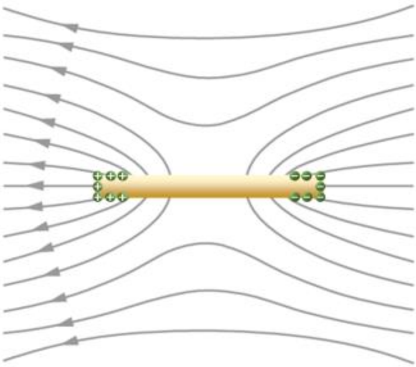
In frame S:

Two opposite sides of the rod would acquire some charges. This charge distribution causes an electric field outside the rod, as well as inside.



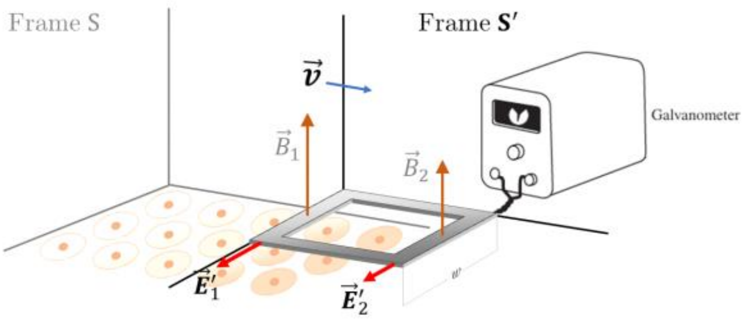
In frame S′:

The redistribution of charge on the surface of the rod so as to make the electric field zero inside.



The charge distribution seen in frame S is the same as that seen in S′. The electric fields differ because the field in frame S is that of the surface charge distribution alone, while the electric field in S′ is the field of the surface charge distribution plus the uniform electric field that exists in S′.

(2) Suppose that the field $\vec{B}$ in the frame S, though constant in time, is not uniform in space.



In frame S:

Evaluate the line integral of $\vec{f}$, taken around the whole loop (counterclockwise as viewed from above).

$$\oint_C \vec{f} \cdot d\vec{s} = qvB_1w - qvB_2w = qv(B_1 - B_2)w$$

The electromotive force that causes charge to circulate around a closed path is

$$\mathcal{E} = \frac{1}{q}\oint_{\mathcal{C}} \vec{f} \cdot d\vec{s} = v(B_1 - B_2)w$$

⇒ The galvanometer deflects.

OS:

> Since $\nabla \times \vec{E} = 0$, for an electrostatic field, such a field cannot cause a charge to circulate around a closed path. Therefore, an EMF must be *nonelectrostatic* in origin.

In frame S′:

According to equations in 5-5-A(a), there will be an electric field in frame S′:

$$\left.\begin{array}{ll} \vec{E}'_{\parallel} = 0, & \vec{E}'_{\perp} = \gamma\vec{v} \times \vec{B}_{\perp} \\ \vec{B}'_{\parallel} = \vec{B}_{\parallel}, & \vec{B}'_{\perp} = \gamma\vec{B}_{\perp} \end{array}\right\} \Rightarrow \vec{E}'_{\perp} = \vec{v} \times \vec{B}'_{\perp} \Rightarrow \vec{E}' = \vec{v} \times \vec{B}'$$

$\Rightarrow \vec{E}'_1 = \vec{v} \times \vec{B}'_1$ and $\vec{E}'_2 = \vec{v} \times \vec{B}'_2$

The line integral of $\vec{E}'$ around any closed path in frame S′ is

$$\oint_{\mathcal{C}} \vec{E}' \cdot d\vec{s}' = v(B'_1 - B'_2)w$$

The electromotive force that causes charge to circulate around a closed path is

$$\mathcal{E}' = \oint_{\mathcal{C}} \vec{E}' \cdot d\vec{s}' = v(B'_1 - B'_2)w$$

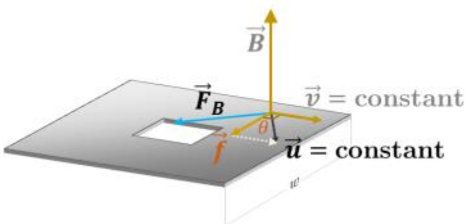
⇒ The galvanometer deflects.

⇒ The EMF is a Lorentz invariance.

EXAMPLES:

1. Since the electromotive force is the work done per unit charge moving around a circuit, thus $\vec{f}$ does work. However, $\vec{f} = q\vec{v} \times \vec{B}$ is a magnetic force and magnetic forces do no work. What is doing work?

   ANSWER:



$\vec{F}_B = q\vec{u} \times \vec{B}$

Since $\vec{F}_B \perp \vec{u}$, i.e., $\vec{F}_B$ is perpendicular to the motion of the charge, the magnetic force does no work.
Assuming that the current is steady and the charge is not accelerating, the net force on it equals zero, i.e., $\vec{v}$ = constant and $\vec{u}$ = constant.

$\vec{F}_{B\parallel} = -\vec{F} = -qu_{\perp}B\hat{v}$ ····· the pulling force

$\vec{F}_{B\perp} = \vec{f} = qu_{\parallel}B$

The work done by $\vec{f}$:

$$\int_a^b \vec{f} \cdot d\vec{s} = \int_a^b \vec{F}_{B\perp} \cdot d\vec{s} = qu_{\parallel}Bw = qvBw$$

The work done by $\vec{F}$:

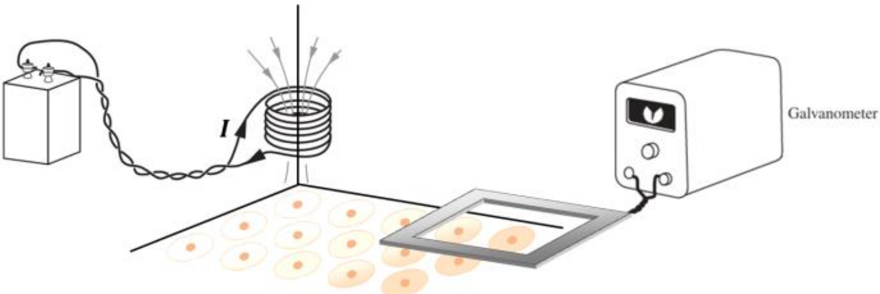
$$\begin{aligned}\int_a^b \vec{F} \cdot d\vec{s} &= -\int_a^b \vec{F}_{B\parallel} \cdot d\vec{s} \\ &= qu_{\perp}B \cdot w\tan\theta \\ &= quBw\sin\theta \\ &= qvBw\end{aligned}$$

This result confirms that the work done by $\vec{f}$ is equals the work done by $\vec{F}$.

The EMF is generated by $\vec{f}$:

$$\mathcal{E} = \frac{1}{q}\oint_{\mathcal{C}} \vec{f} \cdot d\vec{s} = \frac{1}{q}\int_a^b \vec{f} \cdot d\vec{s} = vBw$$

(3) Vary the current I in the coil

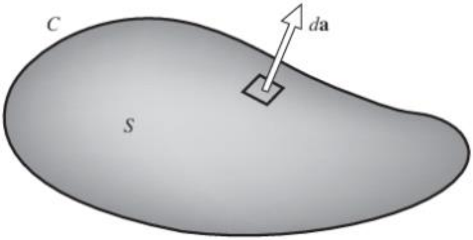


⇒ The galvanometer deflects.

(4) In 1831 Michael Faraday reported on a series of experiments. These experiments were the first to demonstrate a fundamental fact that the current is a result of an EMF induced by a changing magnetic field.

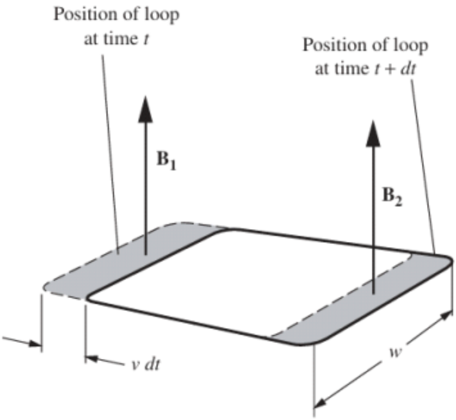
B. FLUX RULE

(1) Define the flux of $\vec{B}$ through the loop, i.e., the surface integral of $\vec{B}$ over a surface $\mathcal{S}$ that has the closed curve $\mathcal{C}$:



$$\Phi = \int_{\mathcal{S}} d\Phi = \int_{\mathcal{S}} \vec{B} \cdot d\vec{a}$$

For a rectangular loop, as the loop moves:



In frame S:

The flux is lost at the left, while the flux is gained at the right:

$$d\Phi = -B_1 wvdt + B_2 wvdt = -(B_1 - B_2)wvdt$$

The electromotive force can be expressed as

$$\mathcal{E} = -\frac{d\Phi}{dt}$$ ⋯⋯ an arbitrary convention about sign

In frame S′:

Stationary loop with the field source moving

The flux is lost or gained at either end of the loop, in a time interval dt'. We get

$$d\Phi' = -B_1' wvdt' + B_2' wvdt' = -(B_1' - B_2')wvdt'$$

The electromotive force can be expressed as

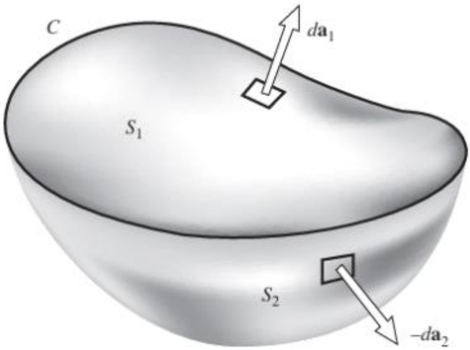
$$\mathcal{E}' = -\frac{d\Phi'}{dt'}$$ ⋯⋯ an arbitrary convention about sign

(2) Flux rule:

If the magnetic field in a given frame is constant in time, then for a loop of any shape moving in any manner, the emf $\mathcal{E}$ around the loop is related to the magnetic flux through the loop by

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

PROOF:



If $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ is any closed surface and $\mathcal{V}$ is the volume inside it, according to Gauss's divergence theorem, we have

$$\oint_{\mathcal{S}} \vec{B} \cdot d\vec{a} = \int_{\mathcal{V}} \nabla \cdot \vec{B}\, d\tau = 0$$

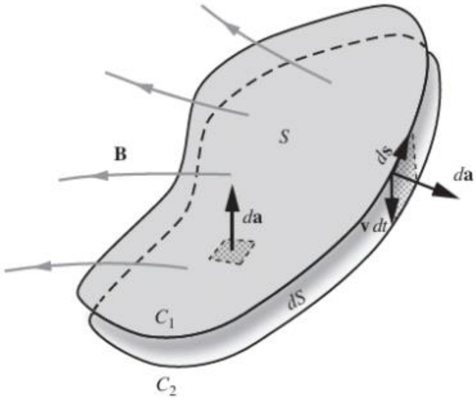
and

$$\oint_{\mathcal{S}} \vec{B} \cdot d\vec{a} = \int_{\mathcal{S}_1} \vec{B} \cdot d\vec{a}_1 + \int_{\mathcal{S}_2} \vec{B} \cdot (-d\vec{a}_2)$$

$$\Rightarrow \int_{\mathcal{S}_1} \vec{B} \cdot d\vec{a}_1 = \int_{\mathcal{S}_2} \vec{B} \cdot d\vec{a}_2$$

This shows that it doesn't matter which surface we use to compute the flux through $\mathcal{C}$.

The loop moves from position $\mathcal{C}_1$ to position $\mathcal{C}_2$ in time dt.



$$\Phi(t+dt)=\int_{\mathcal{S}+d\mathcal{S}}\vec{B}\cdot d\vec{a}=\Phi(t)+\int_{d\mathcal{S}}\vec{B}\cdot d\vec{a}$$

Since

$$d\vec{a}=\vec{v}dt\times d\vec{s}$$

So the integral over the surface $d\mathcal{S}$ can be written as an integral around the path $\mathcal{C}$, in this way:

$$d\Phi=\int_{d\mathcal{S}}\vec{B}\cdot d\vec{a}=\int_{\mathcal{C}}\vec{B}\cdot\left(\vec{v}dt\times d\vec{s}\right)=\int_{\mathcal{C}}\vec{B}\cdot\left(\vec{v}\times d\vec{s}\right)dt$$

Since the scalar triple-product can be rewritten:

$$\vec{B}\cdot(\vec{v}\times d\vec{s})=-\left(\vec{v}\times\vec{B}\right)\cdot d\vec{s}$$

we thus have

$$\frac{d\Phi}{dt}=-\int_{\mathcal{C}}\left(\vec{v}\times\vec{B}\right)\cdot d\vec{s}=-\frac{1}{q}\int_{\mathcal{C}}\vec{f}\cdot d\vec{s}=-\mathcal{E}$$

■

C. UNIVERSAL LAW OF INDUCTION

(1) In frame S, if $\mathcal{C}$ is some closed curve, $\mathcal{S}$ is a surface spanning $\mathcal{C}$; and if $\vec{B}$ is the magnetic field measured in frame S at any time t, then we obtain Faraday's law of induction:

$$\mathcal{E}=\oint_{\mathcal{C}}\vec{E}\cdot d\vec{s}=-\frac{d\Phi}{dt}=-\frac{d}{dt}\int_{\mathcal{S}}\vec{B}\cdot d\vec{a}$$

Using Stokes' theorem:

$$\oint_{\mathcal{C}}\vec{E}\cdot d\vec{s}=\int_{\mathcal{S}}\left(\nabla\times\vec{E}\right)\cdot d\vec{a}$$

We obtain the differential form of Faraday's law as

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

(2) From Faraday's law, we have

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} + \frac{\partial}{\partial t}\nabla \times \vec{A} = \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

Since the curl of gradient of a scalar function f is zero,

$$\nabla \times (\nabla f) = 0$$

thus, the equation above can be solved generally by putting the bracket equal to a gradient of a scalar function $-\nabla\varphi$ which gives the result for electric field in terms of both scalar and vector potentials, i.e.,

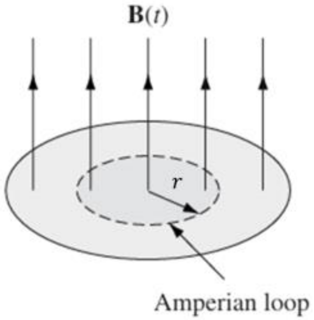
$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla\varphi \Rightarrow \vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}$$

In the absence of vector potential, the electric field is an electrostatic field,

$$\vec{E} = -\nabla\varphi$$

EXAMPLES:

1. A uniform magnetic field $\vec{B}(t)$, pointing straight up, fills the shaded circular region. If $\vec{B}$ is changing with time, what is the in duced electric field?



ANSWER:

- Method I:

$\vec{E}$ points in the circumferential direction,

$$\oint_{\mathcal{C}} \vec{E} \cdot d\vec{s} = -\frac{d}{dt}\int_{\mathcal{S}} \vec{B} \cdot d\vec{a} = -\frac{d}{dt}\left(\pi r^2 B(t)\right) = -\pi r^2 \frac{dB(t)}{dt}$$

Therefore

$$E \cdot 2\pi r = -\pi r^2 \frac{dB(t)}{dt} \Rightarrow \vec{E} = -\frac{r}{2}\frac{dB(t)}{dt}\hat{\phi}$$

- Method II:

$$\vec{B}(t) = B(t)\hat{z} = \nabla \times \vec{A}(t) \Rightarrow \vec{A}(t) = \frac{1}{2}B(t)r\hat{\phi}$$

Verify:

$$\nabla \times \vec{A} = \begin{vmatrix} \frac{1}{r}\hat{r} & \hat{\phi} & \frac{1}{r}\hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & r\frac{B(t)r}{2} & 0 \end{vmatrix} = B(t)\hat{z}$$

Thus, we obtain

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{r}{2}\frac{dB(t)}{dt}\hat{\phi}$$

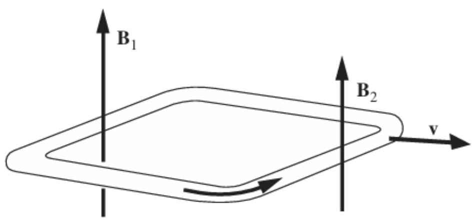
(3) Lenz assures us that the induced EMF will be in the correct direction to cause such a current.

Lenz' law:

The direction of the induced electromotive force is such that the induced current creates a magnetic field that opposes the change in flux.

EXAMPLES:

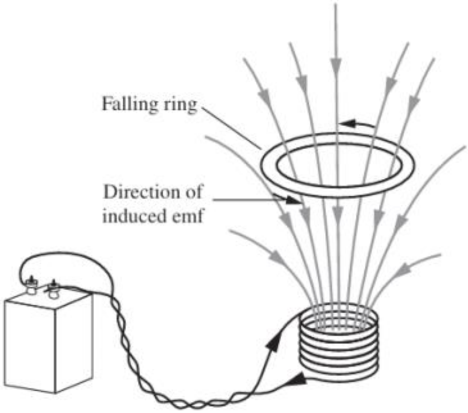
1. The flux through the loop is upward and is decreasing in magnitude as time goes on.



ANSWER:

An electromotive force would tend to drive a positive charge around the loop in a counterclockwise direction. This current itself would create some flux through the loop in a direction to counteract the assumed flux change.

2. As the ring falls, the downward flux through the ring is increasing.



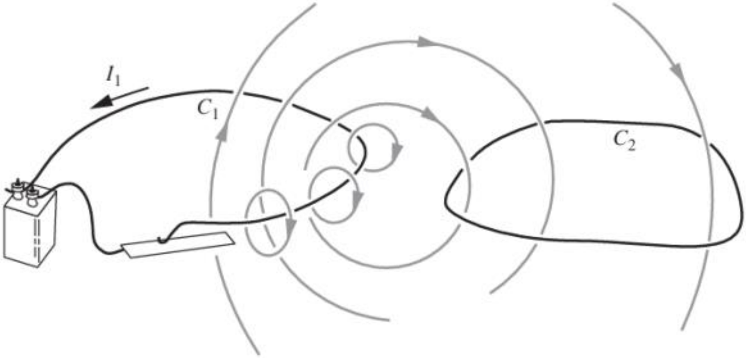
ANSWER:

The induced EMF will be in the direction in which current must flow to produce upward flux through the ring.

7-3 Inductance and RL Circuit

A. MUTUAL INDUCTANCE

(1) Current I_1 in loop $\mathcal{C}_1$ causes a certain flux Φ_{21} through loop $\mathcal{C}_2$.



$$\Phi_{21} = \int_{\mathcal{S}_2} \vec{B}_1 \cdot d\vec{a}_2$$

Since Φ_{21} will be proportional to I_1, thus we define

$$\frac{\Phi_{21}}{I_1} = \text{constant} \equiv M_{21}$$

There will be an electromotive force induced in circuit $\mathcal{C}_2$, of magnitude

$$\mathcal{E}_{21} = -\frac{d\Phi_{21}}{dt} = -M_{21}\frac{dI_1}{dt}$$

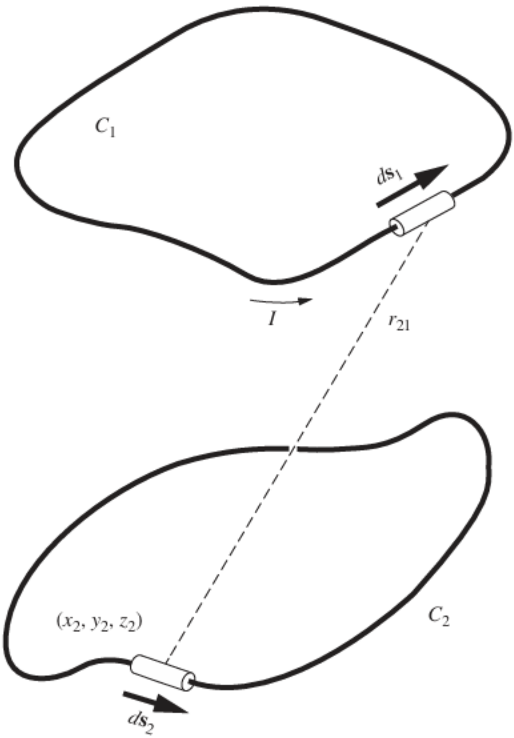
We call the constant M_{21} the coefficient of mutual inductance.

(2) Reciprocity theorem

For any two circuits, we have

$$M_{12} = M_{21}$$

PROOF:



Consider current I_1 in loop $\mathcal{C}_1$ causes a certain flux Φ_{21} through loop $\mathcal{C}_2$:

$$\Phi_{21} = \int_{\mathcal{S}_2} \vec{B}_1 \cdot d\vec{a}_2 = \int_{\mathcal{S}_2} \left(\nabla \times \vec{A}_1\right) \cdot d\vec{a}_2 = \oint_{\mathcal{C}_2} \vec{A}_1 \cdot d\vec{s}_2$$

Since

$$\vec{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{\mathcal{C}_1} \frac{d\vec{s}_1}{r_{21}}$$

thus, we obtain

$$\Phi_{21} = \frac{\mu_0 I_1}{4\pi} \oint_{\mathcal{C}_1} \frac{d\vec{s}_1}{r_{21}} \oint_{\mathcal{C}_2} d\vec{s}_2 = \frac{\mu_0 I_1}{4\pi} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{d\vec{s}_2 \cdot d\vec{s}_1}{r_{21}}$$

Similarly, current I_2 in loop $\mathcal{C}_2$ causes a certain flux Φ_{12} through loop $\mathcal{C}_1$:

$$\Phi_{12} = \frac{\mu_0 I_2}{4\pi} \oint_{\mathcal{C}_2} \oint_{\mathcal{C}_1} \frac{d\vec{s}_1 \cdot d\vec{s}_2}{r_{12}}$$

The mutual instances are respectively

$$M_{21} = \frac{\Phi_{21}}{I_1} = \frac{\mu_0}{4\pi} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{d\vec{s}_2 \cdot d\vec{s}_1}{r_{21}}$$

$$M_{12} = \frac{\Phi_{12}}{I_2} = \frac{\mu_0}{4\pi} \oint_{\mathcal{C}_2} \oint_{\mathcal{C}_1} \frac{d\vec{s}_1 \cdot d\vec{s}_2}{r_{12}}$$

Since

$$r_{12} = r_{12} = r$$

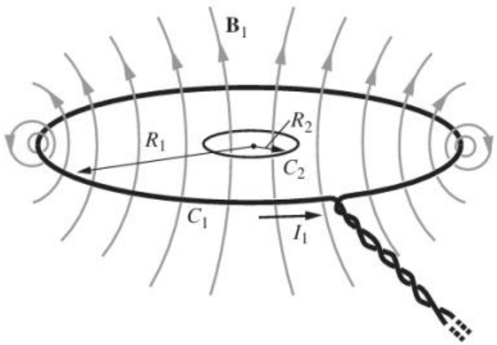
we obtain

$$M_{12} = \frac{\mu_0}{4\pi} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{d\vec{s}_1 \cdot d\vec{s}_2}{r} = M_{21}$$

■

EXAMPLES:

1. Two coplanar, concentric rings: a small ring $\mathcal{C}_2$ and a much larger ring $\mathcal{C}_1$. Assuming $R_2 \ll R_1$, what is the mutual inductance M_{21}?



ANSWER:

At the center of $\mathcal{C}_1$, with I_1 flowing, the field B_1 is given as

$$B_1 = \frac{\mu_0 I_1}{2R_1}$$

The flux through the small ring is then

$$\Phi_{21} = \int_{\mathcal{S}_2} \vec{B}_1 \cdot d\vec{a}_2 = \frac{\mu_0 I_1}{2R_1}(\pi R_2^2) = \frac{\mu_0 \pi I_1 R_2^2}{2R_1}$$

The mutual inductance M_{21} is therefore

$$M_{21} = \frac{\Phi_{21}}{I_1} = \frac{\mu_0 \pi R_2^2}{2R_1}$$

2. If the circuit $\mathcal{C}_1$ consisted of N_1 turns of wire instead of a single ring, and the small loop $\mathcal{C}_2$ consisted of N_2 turns. What is the mutual inductance?

ANSWER:

Since the electromotive force in each turn in $\mathcal{C}_2$ would add to that in the next, making the total electromotive force in that circuit N_2 times that of a single turn. Thus, we have

$$\Phi_{21} = \int_{S_2} \vec{B}_1 \cdot d\vec{a}_2 = \frac{\mu_0 N_1 I_1}{2R_1}\left(N_2 \pi R_2^2\right) = \frac{\mu_0 \pi N_1 N_2 I_1 R_2^2}{2R_1}$$

The mutual inductance is

$$M_{21} = \frac{\Phi_{21}}{I_1} = \frac{\mu_0 \pi N_1 N_2 R_2^2}{2R_1}$$

B. SELF-INDUCTANCE AND RL CIRCUIT

(1) If there is a flux through circuit 1 of the field $\vec{B}_1$ due to the current I_1 in circuit 1, we can define the self-inductance,

$$\frac{\Phi_{11}}{I_1} = \text{constant} \equiv L_1$$

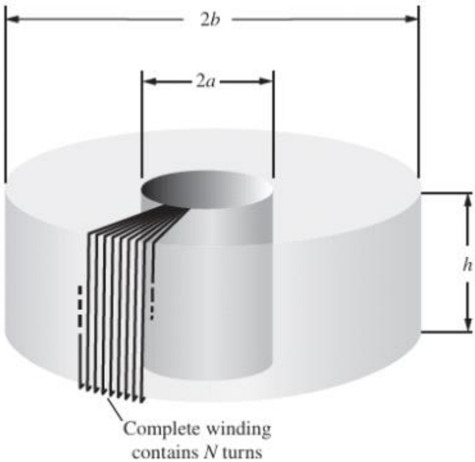
There will be an electromotive force induced in circuit $\mathcal{C}_1$, of magnitude

$$\mathcal{E} = -\frac{d\Phi_{11}}{dt} = -L_1 \frac{dI_1}{dt}$$

The constant L_1 is called the self-inductance of the circuit.

EXAMPLES:

1. Consider a rectangular toroidal coil. What is the self inductance L?



ANSWER:

A current I flowing in the coil of N turns. At a radial distance r from the axis of the coil, the field is given by

$$\oint_{\mathcal{C}} \vec{B} \cdot d\vec{s} = \mu_0 NI \Rightarrow B2\pi r = \mu_0 NI \Rightarrow B = \frac{\mu_0 NI}{2\pi r}$$

The total flux through the coil is the integral of this field over the cross section of the coil:

$$\Phi = N \int_{\mathcal{S}} \vec{B} \cdot d\vec{a} = N \int_a^b \frac{\mu_0 NI}{2\pi r} h dr = \frac{\mu_0 N^2 Ih}{2\pi} \ln\left(\frac{b}{a}\right)$$

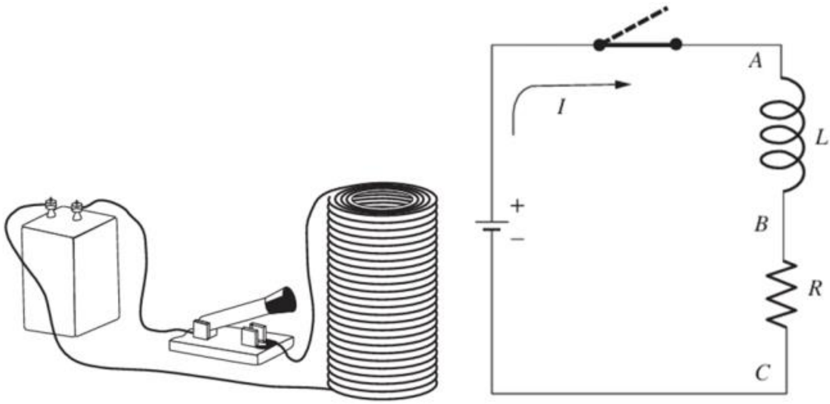
The induced electromotive force is

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right) \frac{dI}{dt}$$

The self-inductance of this coil is given by

$$L = \frac{\Phi}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$$

(2) A simple circuit with inductance L and resistance R.



Electromotive force view:

If the current I in the circuit is changing at the rate $\frac{dI}{dt}$, an electromotive force $L\frac{dI}{dt}$ will be induced, in a direction to oppose the change. The net electromotive force drives the current I through the resistor R is

$$\mathcal{E} - L\frac{dI}{dt} = IR$$

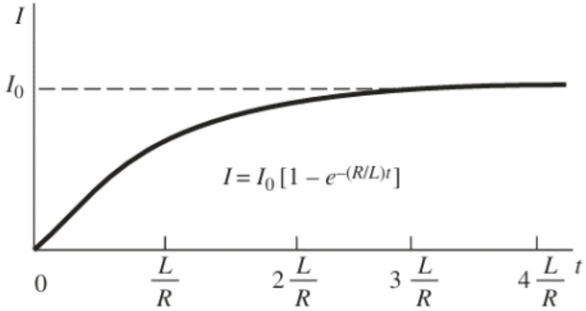
Potential difference view:

The potential difference between the battery terminals,

$$\mathcal{E} = L\frac{dI}{dt} + IR$$

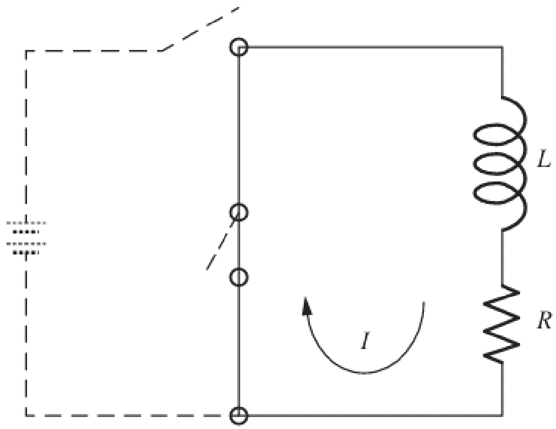
(3) If the switch is closed at $t = 0$, then we find the solution of the equation above as

$$I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-(R/L)t}\right) = I_0\left(1 - e^{-(R/L)t}\right)$$



The current approaching its asymptotic value I_0 exponentially. The "time constant" of this circuit is the quantity L/R.

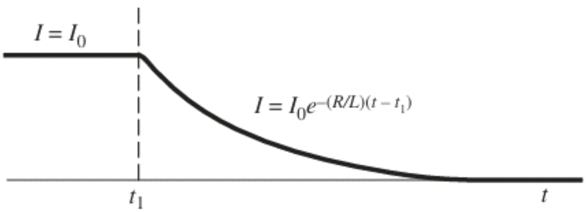
(4) If we open the switch after the current I_0 has been established, we have a circuit described as



$$0 = L\frac{dI}{dt} + IR$$

The solution is the simple exponential decay function

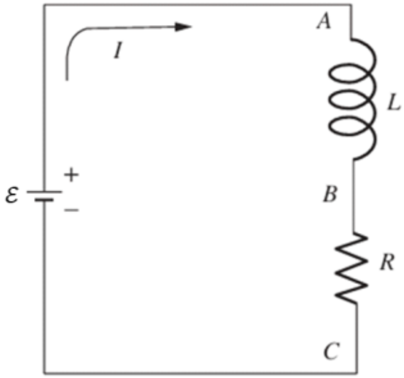
$$I(t) = I_0 e^{-(R/L)(t-t_1)}$$



7-4 Magnetic Energy

A. ENERGY STORED IN THE MAGNETIC FIELD

(1) In a simple circuit, the power supplied by the battery is



$$I\mathcal{E} = IL\frac{dI}{dt} + I^2R$$

The first term is the rate of increase of the magnetic energy. The second term is the power dissipated as heat.
Thus, if U is the stored magnetic energy,

$$\frac{dU}{dt} = IL\frac{dI}{dt} = I\frac{d\Phi}{dt} \Rightarrow dU = LIdI = Id\Phi$$

Clearly, $U = 0$ when $I = 0$, then

$$U = \int dU = \int LI\,dI = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi$$

EXAMPLES:

1. If we have two circuits a and b. The magnetic energy stored is
   ANSWER:

$$\begin{aligned} U_a + U_b &= \frac{1}{2}\left(I_a\Phi_a + I_b\Phi_b\right) \\ &= \frac{1}{2}\left(I_a\left(L_aI_a + MI_b\right) + I_b\left(L_bI_b + MI_a\right)\right) \\ &= \frac{1}{2}L_aI_a^2 + \frac{1}{2}L_bI_b^2 + MI_aI_b \end{aligned}$$

(2) The magnetic energy U expressed in terms of $\vec{A}$ and $\vec{J}$.

$$\frac{dU}{dt}=I\frac{d\Phi}{dt}=I\frac{d}{dt}\int_{\mathcal{S}}\vec{B}\cdot d\vec{a}=I\frac{d}{dt}\int_{\mathcal{S}}\left(\nabla\times\vec{A}\right)\cdot d\vec{a}=I\frac{d}{dt}\oint_{\mathcal{C}}\vec{A}\cdot d\vec{s}$$

Since

$$Id\vec{s}=Jda'd\vec{s}=\vec{J}d\tau$$

where da' is the cross-section area of the wire, we have

$$I\frac{d\Phi}{dt}=\frac{d}{dt}\oint_{\mathcal{C}}\vec{A}\cdot Id\vec{s}=\int_{\mathcal{V}}\vec{J}\cdot\frac{\partial\vec{A}}{\partial t}d\tau$$

where $\mathcal{C}$ is the loop of the wire, and $\mathcal{V}$ is the volume of the wire.

Since

$$\frac{d}{dt}\int_{\mathcal{V}}\vec{J}\cdot\vec{A}\,d\tau=\underbrace{\int_{\mathcal{V}}\vec{J}\cdot\frac{\partial\vec{A}}{\partial t}d\tau}_{=I\frac{d\Phi}{dt}}+\int_{\mathcal{V}}\frac{\partial\vec{J}}{\partial t}\cdot\vec{A}\,d\tau$$

Similarly, we can obtain

$$\Phi\frac{dI}{dt}=\frac{d}{dt}\oint_{\mathcal{C}}I\vec{A}\cdot d\vec{s}=\int_{\mathcal{V}}\frac{\partial\vec{J}}{\partial t}\cdot\vec{A}\,d\tau$$

Thus, we get

$$\frac{d}{dt}\int_{\mathcal{V}}\vec{J}\cdot\vec{A}\,d\tau=\underbrace{\int_{\mathcal{V}}\vec{J}\cdot\frac{\partial\vec{A}}{\partial t}d\tau}_{=I\frac{d\Phi}{dt}}+\underbrace{\int_{\mathcal{V}}\frac{\partial\vec{J}}{\partial t}\cdot\vec{A}\,d\tau}_{=\Phi\frac{dI}{dt}}$$

Since

$$IL\frac{dI}{dt}=I\frac{d\Phi}{dt}\Rightarrow\Phi\frac{dI}{dt}=I\frac{d\Phi}{dt}$$

we have

$$\frac{d}{dt}\int_{\mathcal{V}}\vec{J}\cdot\vec{A}\,d\tau=2\int_{\mathcal{V}}\vec{J}\cdot\frac{\partial\vec{A}}{\partial t}d\tau\Rightarrow\int_{\mathcal{V}}\vec{J}\cdot\frac{\partial\vec{A}}{\partial t}d\tau=\frac{1}{2}\frac{d}{dt}\int_{\mathcal{V}}\vec{J}\cdot\vec{A}\,d\tau$$

Thus, we obtain

$$\frac{dU}{dt}=\frac{1}{2}\frac{d}{dt}\int_{\mathcal{V}}\vec{J}\cdot\vec{A}\,d\tau\Rightarrow U=\frac{1}{2}\int_{\mathcal{V}}\left(\vec{A}\cdot\vec{J}\right)d\tau$$

(3) The magnetic energy U expressed in terms of $\vec{B}$.

Using Ampère's law,

$$U=\frac{1}{2}\int_{\mathcal{V}}\left(\vec{A}\cdot\vec{J}\right)d\tau=\frac{1}{2\mu_0}\int_{\mathcal{V}}\vec{A}\cdot\left(\nabla\times\vec{B}\right)d\tau$$

Using vector product rules of ∇:

$$\nabla\cdot\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot\left(\nabla\times\vec{A}\right)-\vec{A}\cdot\left(\nabla\times\vec{B}\right)=\vec{B}\cdot\vec{B}-\vec{A}\cdot\left(\nabla\times\vec{B}\right)$$

Thus we obtain

$$U = \frac{1}{2\mu_0}\left[\int_{\mathcal{V}} B^2\,d\tau - \int_{\mathcal{V}} \nabla\cdot\left(\vec{A}\times\vec{B}\right)d\tau\right]$$
$$= \frac{1}{2\mu_0}\left[\int_{\mathcal{V}} B^2\,d\tau - \oint_{\mathcal{S}} \left(\vec{A}\times\vec{B}\right)\cdot d\vec{a}\right]$$

If we take the surface $\mathcal{S}$ to infinity, then

$$\underbrace{\vec{A}}_{\sim\frac{1}{r}} \times \underbrace{\vec{B}}_{\sim\frac{1}{r^2}} \propto \frac{1}{r^3}$$

Since the area of the surface $\mathcal{S}$ grows only like r^2, the surface integral therefore goes to zero as $r\to\infty$. Thus, we obtain

$$U = \frac{1}{2\mu_0}\int B^2\,d\tau = \frac{1}{2}LI^2$$

EXAMPLES:

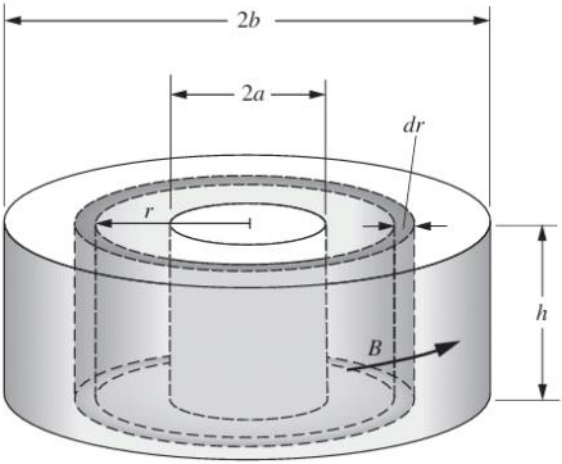
1. Magnetic fields do no work. How the energy is stored in the magnetic field?

   ANSWER:

   Producing a magnetic field, where previously there was none, requires changing the field, and a changing $\vec{B}$-field, according to Faraday, induces an electric field. The latter, of course, can do work.

   In the beginning, there is no $\vec{E}$, and at the end there is no $\vec{E}$; but in between, while $\vec{B}$ is building up, there is an $\vec{E}$, and it is against this that the work is done.

2. Calculation of energy stored in the magnetic field of the toroidal coil.



ANSWER:

The magnetic field strength B, with current I flowing, was given by

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_0 NI \Rightarrow B2\pi r = \mu_0 NI \Rightarrow B = \frac{\mu_0 NI}{2\pi r}$$

Since

$$\frac{1}{2\mu_0}\int B^2\, d\tau = \frac{1}{2\mu_0}\int_a^b \left(\frac{\mu_0 NI}{2\pi r}\right)^2 h2\pi r dr = \frac{\mu_0 N^2 h I^2}{4\pi}\ln\left(\frac{b}{a}\right) = \frac{1}{2}LI^2$$

We obtain the self-inductance

$$L = \frac{\mu_0 N^2 h}{2\pi}\ln\left(\frac{b}{a}\right)$$